THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Suggested Solution to Assignment 3

1. Recall the line separation property with another formulation:

   If B is a point and l is a line passing through the point B, then we can define a equivalence relation on the points of $l \backslash \{B\}$ such that $A \sim C$ if and only if the line segment AC does not contain B, i.e. $A * B * C$ is not true.

   Furthermore, there are only two equivalence classes, hence we say A and C are said to be on the same side of B if $A \sim C$, otherwise they are said to be on opposite side of B.

   (a) Considering the above equivalence relation with a fixed point B. By the assumption that $A * B * C$, we know that $A \nsim C$. Furthermore, by axiom **B3**, $B * C * D$ implies that $C * B * D$ is not true, i.e. $C \sim D$. Therefore, $A \nsim D$ and so $A * B * D$.

   Note that $A * B * C$ and $B * C * D$ imply $D * C * B$ and $C * B * A$ by axiom **B1**. By the above argument, we have $D * C * A$, as well as $A * C * D$ by axiom **B1** again.

   (b) Considering the above equivalence relation with a fixed point B. By the assumption that $A * B * D$, we know that $A \nsim D$. Furthermore, by axiom **B3**, $B * C * D$ implies that $C * B * D$ is not true, i.e. $C \sim D$. Therefore, $A \nsim C$ and so $A * B * C$.

   Then, by the above and (a), $A * B * C$ and $B * C * D$ implies $A * C * D$.

2. Let l be a line. By axiom **I2**, there exist two distinct points B_1 and B_2 lying on l.

   By axiom **B2**, there exists B_3 such that $B_1 * B_2 * B_3$. Repeating this argument, there exists an infinite sequence of points B_n on l such that $B_n * B_{n+1} * B_{n+2}$, for $n = 1, 2, 3, \ldots$.

   Therefore, the line l has infintely many distinct points.

3. Let A and B be two distinct points.

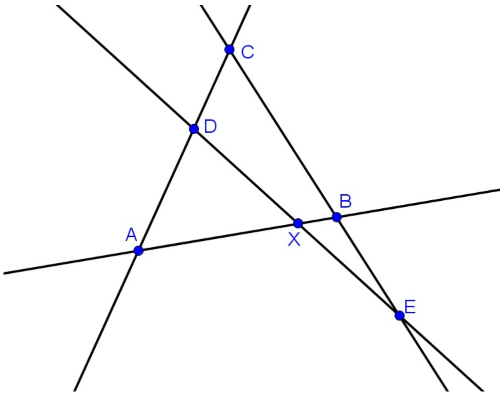
   There exists a point D that does not lying on the line passing through A, B.†

   By axiom **B2**, there exists C such that $A * D * C$ and there exists F such that $C * B * E$.

   Then consider the line l passing through D and E (axiom **I1**), firstly $A, B, C \notin l$.‡

   Also, l contains D with the property $A * D * C$, but it does not any point lying between B and C since $C * B * E$.

By axiom **B4**, the line l must contain a point X such that $A * X * B$.



4. Let A, B and C be three noncollinear points.

   By using the result in the previous question, there exist points D, E such that $A * D * B$ and $A * E * C$.

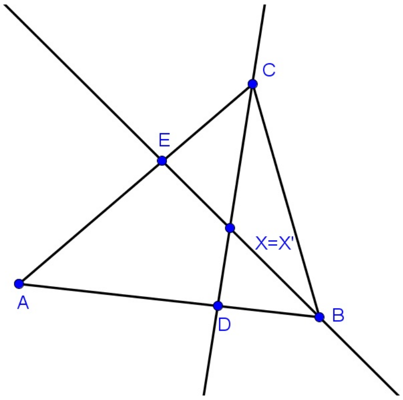
   In $\triangle ABE$, consider the line l_{CD} passing through the point C and D (axiom **I1**), it contains D with the property $A * D * B$, but it does not any point lying between A and E since $A * E * C$.

   By axiom **B4**, the line l_{CD} must contain a point X such that $B * X * E$.

   Similarly, in $\triangle ACD$, consider the line l_{BE} passing through the point B and E (axiom **I1**), it contains E with the property $A * E * C$, but it does not any point lying between A and D since $A * D * B$.

   By axiom **B4**, the line l_{BE} must contain a point X' such that $C * X' * D$.

   Therefore, X and X' are points that lie on both l_{CD} and l_{BE} which forces that $X = X'$. Also, we have $B * X * E$ and $C * X * D$.



   By using crossbar theorem, the ray r_{AX} contains a point F such that $B * F * C$. We claim that $A * X * F$.

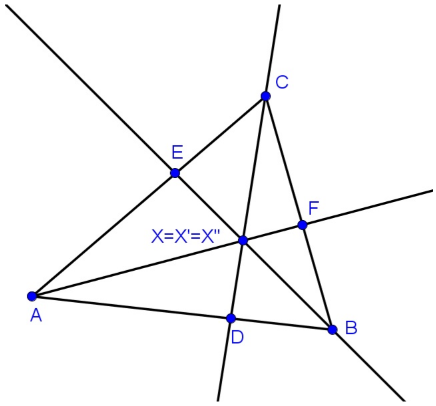
   Once again, in $\triangle ABF$, consider the line l_{CD} passing through the point C and D (axiom **I1**), it contains D with the property $A * D * B$, but it does not any point lying between B and F since $B * F * C$.

   By axiom **B4**, the line l_{CD} must contain a point X'' such that $A * X'' * F$.

Therefore, X and X'' are points that lie on both l_{CD} and l_{AF} which forces that $X = X''$. Also, we have $A * X * F$.

Note that B and X are on the same side of the line l_{AC} since $A * X * E$, also C and X are on the same side of the line l_{AB} since $C * X * D$. Therefore, X is an interior point of $\angle BAC$.

Similarly, we can show that X is an interior point of $\angle ABC$ and $\angle BCA$. As a result, X is an interior point of the triangle ABC.



5. (a) Let Γ be a circle with center O and radius OA.

   Let l be any line passing through O. By the line separation property, $l \backslash \{O\}$ can be divided into two nonempty disjoint subsets S_1 and S_2. Also, $r_1 = S_1 \cup \{O\}$ and $r_2 = S_2 \cup \{O\}$ are two rays with the same vertex O.

   By axiom **C1**, there exists a unique B_i on the ray r_i such that $OA \cong OB_i$, for $i = 1, 2$.

   Therefore, $l \cap \Gamma = (r_1 \cup r_2) \cap = (r_1 \cap \Gamma) \cup (r_2 \cap \Gamma) = \{B_1, B_2\}$.

   (b) Let Γ be a circle.

   There exists a line l which does not contain O.[†]

   By question 2, l contains an infinite sequence of points B_n.

   Then, we have an infinite sequence of lines l_{OB_n} such that they all pass through the point O. Note that if $i \neq j$, $l_{OB_i} \neq l_{OB_j}$ and $l_{OB_i} \cap l_{OB_j} = \{O\}$.[‡] By (a), each line l_{OB_i} contains two points of the circle Γ while these two points must not lie on another line l_{OB_j} for $i \neq j$.

   Therefore, a circle has infinitely many points.

6. Let $A = (0, 1)$ and $B = (1, 2)$. Then $d(A, B) = \sqrt{(1-0)^2 + (2-1)^2} = \sqrt{2}$.

   Let r be the ray $\{(x, 0) : x \in \mathbb{Q} \text{ and } x > 0\}$ which has vertex O. Then, for any point $C = (x, 0)$ on the ray r, $d(O, C) = x$ which is rational number and it cannot be $\sqrt{2}$.

   Therefore, there exists no C on the ray r such that $AB \cong OC$.

7. (a) Let A and B are distinct points and let $L = d(A, B)$. Since A and B are distinct points, $L > 0$.

   Let $C = (c_1, c_2)$ and r is a ray with vertex C. Consider the quadrilateral with vertices $(c_1 + L, c_2)$, $(c_1, c_2 + L)$, $(c_1 - L, c_2)$ and $(c_1, c_2 - L)$. We can see that the distance between any point on that quadrilateral and C is L and the ray r must intersect that quadrilateral exactly at one point D. Therefore, D is the unique point on r such that $AB \cong CD$.

(Remark: The quadrilateral constructed is in fact the circle centered at C with radius L.)

(b) Biscally, we have to show $\cong$ is an equivalance relation, but it simply follows from the fact that equality of real number is an equivalence relation.

(c) Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, $C = (c_1, c_2)$ be three points such that $A * B * C$. We claim that

$$d(A, C) = |a_1 - c_1| = |a_1 - b_1| + |b_1 - c_1| = d(A, B) + d(B, C).$$

By the definition of $A * B * C$, it means that we have either $a_1 * b_1 * c_1$ or $a_2 * b_2 * c_2$ or both, where $a_1 * b_1 * c_1$ means $a_1 < b_1 < c_1$ or $a_1 > b_1 > c_1$ and so on. Note that if $a_1 * b_1 * c_1$, for both cases, we must have $|a_1 - c_1| = |a_1 - b_1| + |b_1 - c_1|$ Therefore, if we have both $a_1 * b_1 * c_1$ and $a_2 * b_2 * c_2$, then we have

$$\begin{aligned} d(A, C) &= |a_1 - c_1| + |a_2 - c_2| \\ &= (|a_1 - b_1| + |b_1 - c_1|) + (|a_2 - b_2| + |b_2 - c_2|) \\ &= (|a_1 - b_1| + |a_2 - b_2|) + (|b_1 - c_1| + |b_2 - c_2|) \\ &= d(A, B) + d(B, C). \end{aligned}$$

If we have $a_1 * b_1 * c_1$ only, since $A * B * C$, we must have $a_2 = b_2 = c_2$ (A, B and C lie on a vertical line). Then we have

$$d(A, C) = |a_1 - c_1| = |a_1 - b_1| + |b_1 - c_1| = d(A, B) + d(B, C)$$

and the above equation holds for the case that $a_2 * b_2 * c_2$ only.

Therefore, if C, D and E are three points such that $D * E * F$ and $AB \cong DE$, $BC \cong EF$. Then

$$d(A, C) = d(A, B) + d(B, C) = d(D, E) + d(E, F) = d(D, F)$$

which shows that $AC \cong DE$.

Therefore, axioms **C1**, **C2** and **C3** hold.

† We have proved this result before.

‡ Need one line argument.